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1995 J. Phys. A: Math. Gen. 28 4313

(http://iopscience.iop.org/0305-4470/28/15/012)

J. Phys. A: Math. Gen. 28 (1995) 4313–4322. Printed in the UK

Toroid moments induced in a spatial charged oscillator by a (time periodic) current density

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Received 7 March 1995

Abstract. The frequency-dependent toroid dipole polarizability $\gamma_{l=1}(\omega)$ which characterizes the linear response of a system to a conduction and/or displacement time-dependent external current is calculated exactly in the tridimensional charged oscillator model. Two frequencies of resonance are obtained. One of them ($\omega_{\rm res} = 3\omega_0$, ω_0 being the frequency of the oscillator) individualizes the toroid dipole polarizability since none of the other (usual) dipole and quadrupole electric and magnetic polarizabilities resonates at this frequency. Comparing the static result $\gamma^{\rm T}_{l=1}(\omega = 0)$ with the static electric and magnetic quadrupole polarizabilities one can see that the toroid effects appear to be of the same order of magnitude as the magnetic ones and that they are very small with respect to the electric effects, but their relative importance increases proportional with $\hbar\omega_0/m_0c^2$ (m_0 being the mass of the oscillator). For elementary particles (particularly at the subhadronic level) induced toroid moments might become predominant.

1. Introduction

In the framework of classical electromagnetism, the discovery of a family of toroid multipole moments [1, 2] which are independent from the other (usual) electric and magnetic ones and cannot be reduced to those [1, 2] was of special importance. The toroid dipole moment (known in the field of elementary particles as Zeldovich's 'anapole') is the first element of this family. The toroid moments are currently being studied in a variety of contexts [1–8].

The *intrinsic* toroid moments of elementary quantum systems are ruled out by invariance under either parity or time reversal. Consequently, the intrinsic toroid moments of such objects should actually be extremely small, being determined by parity or time reversal violating interactions. Nevertheless, the effects related to toroid moments have been theoretically investigated in atomic [3] and nuclear [4] physics. In connection with toroid moments the fact that there is a class of particles, the Majorana fermions, currently occurring in connection with various aspects of neutrino physics (double β-decay, neutrino oscillation, etc) as well as in grand unified and supersymmetric theories, whose possible electromagnetic structure is required (by CPT invariance only) to consist exclusively of toroid moments and distribution [5] (all other usual electric and magnetic moments and distributions are forbidden [5]) must be mentioned.

While the *intrinsic* toroid moments of elementary quantum systems are ruled out by the discrete symmetries, in general there is nothing to prevent the appearance in such systems of induced toroid moments and distribution when external fields are present [6]. The size of such induced toroid moments is measured by toroid polarizabilities [6–8].

0305-4470/95/154313+10$19.50

2. Interaction of the toroid dipole moment with the external field

When a system of charges and currents (specified by $\rho(\boldsymbol{r}, t)$ and $\boldsymbol{j}(\boldsymbol{r}, t)$) interacts with the external electromagnetic field $\boldsymbol{E}^{\text{ext}}$, $\boldsymbol{H}^{\text{ext}}$ described by the potentials $\phi^{\text{ext}}(\boldsymbol{r}, t)$, $\boldsymbol{A}^{\text{ext}}(\boldsymbol{r}, t)$, in the interaction energy

$$W(t) = \int \rho(\boldsymbol{r}, t)\phi^{\text{ext}}(\boldsymbol{r}, t)\, \mathrm{d}^3r - \frac{1}{c}\int \boldsymbol{j}(\boldsymbol{r}, t)\boldsymbol{A}^{\text{ext}}(\boldsymbol{r}, t)\, \mathrm{d}^3r \tag{1}$$

alongside the usual electric and magnetic dipole and higher multipole contributions, there are also contributions from the interaction of the toroid moments with the external field [1, 2]. The toroid dipole term is [1, 2]

$$\begin{aligned} W(t)_{\text{Toroid dipole}} &= -\boldsymbol{T}(t)[\nabla \times \boldsymbol{H}^{\text{ext}}(\boldsymbol{r}, t)]_{r=0,t} \\ &= -\boldsymbol{T}(t)[(4\pi/c)\boldsymbol{J}^{\text{ext}}(\boldsymbol{r}, t) + (1/c)\dot{\boldsymbol{D}}^{\text{ext}}(\boldsymbol{r}, t)]_{r=0,t} \end{aligned} \tag{2}$$

where $\boldsymbol{J}^{\text{ext}}(\boldsymbol{r}, t)$ and $(4\pi)^{-1}\dot{\boldsymbol{D}}^{\text{ext}}(\boldsymbol{r}, t)$ are the external conduction and displacement currents, and

$$\boldsymbol{T}(t) = \frac{1}{10c}\int \{\boldsymbol{r}[\boldsymbol{r} \cdot \boldsymbol{j}(\boldsymbol{r}, t)] - 2r^2\boldsymbol{j}(\boldsymbol{r}, t)\}\, \mathrm{d}^3r \tag{3}$$

is the toroid dipole moment [2].

In the quantum case, as a result of the particular interaction from equation (2), according to the well known non-stationary perturbation rules, a toroid dipole moment (characterized by close toroidal or eight-like currents) will be induced in the system [6] (irrespective of whether or not the system possesses a non-zero intrinsic one). It has the following Fourier components:

$$T_i^{\text{induced}}(\omega) = \sum_j \gamma_{ij}^p(\omega)[\nabla \times \boldsymbol{H}^{\text{ext}}(\omega)]_j \tag{4}$$

where $\gamma_{ij}^p(\omega)$ is the dynamic (i.e. frequency, ω dependent) toroid dipole polarizability [6] (see also [7]) of the quantum system (on the arbitrary state $|p\rangle$ of E_p energy):

$$\begin{aligned} \gamma_{ij}^p(\omega) &= \mathrm{i}\int \mathrm{e}^{\mathrm{i}\omega t}\theta(t)\langle p|[T_i(t), T_j(0)]|p\rangle\, \mathrm{d}t \\ &= \sum_n \left[\frac{\langle p|T_i|n\rangle\langle n|T_j|p\rangle}{E_n - E_p - \hbar\omega - \mathrm{i}\epsilon} + \frac{\langle p|T_j|n\rangle\langle n|T_i|p\rangle}{E_n - E_p + \hbar\omega + \mathrm{i}\epsilon}\right] \end{aligned} \tag{5}$$

(the sum extends over the entire unperturbed spectrum whether discrete or continuous, E_n being the eigenvalues of the unperturbed Hamiltonian).

As shown in [6] the toroid polarizabilities cannot be re-expressed in terms of the usual electric and magnetic multipole polarizabilities, so they are new, independent characteristics of the system.

The purpose of this paper is to compare the toroid dipole polarizability of a quantum system assumed to have an energy spectrum of a tridimensional harmonic oscillator (since this is the most used model in theoretical physics) with the other electric and magnetic polarizabilities which have the same dimensions and are involved with the same importance in the description of many different effects (such as Compton scattering and van der Waals forces).

3. Tridimensional charged oscillator in an external field configuration with $\nabla \times \boldsymbol{H}^{\text{ext}}$ homogeneous

The non-relativistic quantum mechanical Hamiltonian describing the interaction of a spatial charged oscillator with an external field configuration for which $\nabla \times \boldsymbol{H}^{\text{ext}}$ is homogeneous in space ($\boldsymbol{r}$ independent) but generally time-dependent has the form:

$$\mathcal{H}(t) = \frac{1}{2m_0}[\boldsymbol{P} - (q/c)\boldsymbol{A}^{(T)}(\boldsymbol{r}, t)]^2 + \frac{m_0\omega_0^2}{2}r^2 \tag{6}$$

where $\boldsymbol{P} = -i\hbar\nabla$ and q, m_0 denote the charge and the mass of the oscillator, while the vector potential $\boldsymbol{A}^{(T)}(\boldsymbol{r}, t)$ is chosen to be as in [9] (see also [7]):

$$A_i^{(T)}(\boldsymbol{r}, t) = (1/10)(x_i x_j - 2r^2\delta_{ij})[\nabla \times \boldsymbol{H}^{\text{ext}}(t)]_j. \tag{7}$$

The Hamiltonian from equation (6) can be expressed in the form

$$\mathcal{H}(t) = \mathcal{H}_0 + \mathcal{H}_{\text{Toroid dipole}}(t) + \mathcal{H}_{\text{L}}(t) \tag{8}$$

with

$$\mathcal{H}_0 = \frac{\boldsymbol{P}^2}{2m_0} + \frac{m_0\omega_0^2}{2}r^2 \tag{9}$$

$$\mathcal{H}_{\text{Toroid dipole}}(t) = -\frac{q}{m_0 c}\boldsymbol{A}^{(T)}(\boldsymbol{r}, t)\boldsymbol{P} = -[\nabla \times \boldsymbol{H}^{\text{ext}}(t)]\boldsymbol{T} = -\frac{4\pi}{c}\boldsymbol{J}_{\text{Total}}^{\text{ext}}(t)\boldsymbol{T} \tag{10}$$

$$\mathcal{H}_{\text{L}}(t) = \frac{q^2}{200m_0c^2}(-3r^2x_jx_k + 4r^4\delta_{jk})[\nabla \times \boldsymbol{H}^{\text{ext}}(t)]_j[\nabla \times \boldsymbol{H}^{\text{ext}}(t)]_k. \tag{11}$$

$\mathcal{H}_0$ is the unperturbed Hamiltonian for the harmonic oscillator. $\mathcal{H}_{\text{Toroid dipole}}$ and $\mathcal{H}_{\text{L}}$ (a Langevin-type interaction, arising from the term quadratic in $\boldsymbol{A}^{(T)}$ and representing a toroid analogue [9] of the usual part of the Hamiltonian responsible for the Langevin diamagnetism) may be viewed as perturbations. In equation (10) the one-particle operator for the toroid dipole moment, from equation (3) is [2]

$$T_i = \frac{q}{10m_0c}\sum_k(-2r^2\delta_{ik} + x_ix_k)P_k \tag{12}$$

while in equation (10) $\boldsymbol{J}_{\text{Total}}^{\text{ext}}$ denotes the total external current (external conduction and displacement currents).

The exact calculus of the toroid dipole polarizability from equation (5) in the case of an anisotropic charged tridimensional oscillator in an arbitrary state described by quantum numbers $(p_1p_2p_3)$, corresponding to the energy level

$$E_{p_1p_2p_3} = (p_1 + 1/2)\hbar\omega_1 + (p_2 + 1/2)\hbar\omega_2 + (p_3 + 1/2)\hbar\omega_3 \tag{13}$$

together with particular forms of this polarizability will be presented in detail elsewhere [10]. The non-diagonal elements of $\gamma_{ik}^{p_1p_2p_3}(\omega)$ vanish. For an isotropic oscillator ($\omega_1 = \omega_2 = \omega_3 \stackrel{\text{def}}{=} \omega_0$) the expression for $\gamma_{11}^{p_1p_2p_3}(\omega)$ is ($\gamma_{22}^{p_1p_2p_3}(\omega)$ and $\gamma_{33}^{p_1p_2p_3}(\omega)$ can be obtained from $\gamma_{11}^{p_1p_2p_3}(\omega)$ by circular permutation of the indices 1, 2, 3)

$$\gamma_{11}^{p_1p_2p_3}(\omega) = \frac{q^2\hbar^2}{1600m_0^3c^2}\left\{\frac{3[36\epsilon_1^2 + 4(\epsilon_2^2 + \epsilon_3^2) + 16\epsilon_1(\epsilon_2 + \epsilon_3) + 21]}{9\omega_0^2 - \omega^2} + \frac{12\epsilon_1^2 + 28(\epsilon_2^2 + \epsilon_3^2) + 208\epsilon_1(\epsilon_2 + \epsilon_3) + 128\epsilon_2\epsilon_3 - 53}{\omega_0^2 - \omega^2}\right\} \tag{14}$$

with $\epsilon_i = p_i + 1/2$.

The components of the frequency-dependent vector amplitude of the toroid dipole moment induced in the oscillator $\boldsymbol{T}^{\text{induced}}$ will be given by

$$T_i^{\text{induced}}(\omega) = \frac{q^2\hbar^2}{1600 m_0^3 c^2}\left[\frac{3(16\epsilon_i^2 + 4\sum_k \epsilon_k^2 + 16\epsilon\epsilon_i + 21)}{9\omega_0^2 - \omega^2} + \frac{64\epsilon^2 + 80\epsilon\epsilon_i - 96\epsilon_i^2 - 36\sum_k \epsilon_k^2 - 53}{\omega_0^2 - \omega^2}\right](\nabla \times \boldsymbol{H}^{\text{ext}})_i \tag{15}$$

with $\epsilon = \epsilon_1 + \epsilon_2 + \epsilon_3$.

The scalar toroid dipole polarizability

$$\gamma_{l=1}^{p_1p_2p_3}(\omega) = 1/3\sum_i \gamma_{ii}^{p_1p_2p_3}(\omega) \tag{16}$$

takes on the following form:

$$\gamma_{l=1}^{p_1p_2p_3}(\omega) = \frac{q^2\hbar^2}{4800 m_0^3 c^2}\left[\frac{3(16\epsilon^2 + 28\sum_k \epsilon_k^2 + 63)}{9\omega_0^2 - \omega^2} + \frac{272\epsilon^2 - 204\sum_k \epsilon_k^2 - 159}{\omega_0^2 - \omega^2}\right]. \tag{17}$$

For $\omega = 0$ the dynamic tensor and scalar toroid dipole polarizabilities turn in their correspondent static expressions:

$$\gamma_{ii}^{p_1p_2p_3}(\omega = 0) = \frac{q^2\hbar^2}{2400 m_0^3 c^2 \omega_0^2}\left(96\epsilon^2 + 128\epsilon\epsilon_i - 136\epsilon_i^2 - 52\sum_k \epsilon_k^2 - 69\right) \tag{18}$$

$$\gamma_{l=1}^{p_1p_2p_3}(\omega = 0) = \frac{q^2\hbar^2}{7200 m_0^3 c^2 \omega_0^2}\left(416\epsilon^2 - 292\sum_k \epsilon_k^2 - 207\right). \tag{19}$$

These results can be particularized for a ground-state oscillator ($p_1 = p_2 = p_3 = 0$). Because of the spherical symmetry

$$\gamma_{ik}^{000}(\omega) = \gamma_{l=1}^{000}(\omega)\delta_{ik} \tag{20}$$

and the dynamic toroid dipole polarizability of the ground-state oscillator from equation (17) is

$$\gamma_{l=1}^{000}(\omega) = \frac{q^2\hbar^2}{80 m_0^3 c^2}\left(\frac{6}{9\omega_0^2 - \omega^2} + \frac{5}{\omega_0^2 - \omega^2}\right). \tag{21}$$

The static value has the simple form:

$$\gamma_{l=1}^{000}(\omega = 0) = \frac{17 q^2\hbar^2}{240 m_0^3 c^2 \omega_0^2}. \tag{22}$$

The Langevin-type part of the interaction Hamiltonian $\mathcal{H}_{\text{L}}$ from equation (11) restricted to the case in which $\nabla \times \boldsymbol{H}^{\text{ext}}$ is not only homogeneous but also constant in time $[(\nabla \times \boldsymbol{H}^{\text{ext}})^{(c)}]$ induces the energy shift

$$\Delta E_p = \langle p|\mathcal{H}_{\text{L}}|p\rangle = \frac{q^2\hbar^2}{1600 m_0^3 c^2 \omega_0^2}\sum_i\left(32\epsilon^2 - 24\epsilon\epsilon_i - 12\epsilon_i^2 + 16\sum_k \epsilon_k^2 + 27\right)[(\nabla \times \boldsymbol{H}^{\text{ext}})^{(c)}]_i^2. \tag{23}$$

Writing the energy shift as

$$\Delta E_p = -\tfrac{1}{2}\gamma_{ik}^{L,p_1p_2p_3}[(\nabla \times \boldsymbol{H}^{\text{ext}})^{(c)}]_i[(\nabla \times \boldsymbol{H}^{\text{ext}})^{(c)}]_k \tag{24}$$

one finds the following Langevin-type contribution to the toroid dipole tensor polarizability (whose non-zero components are only the diagonal ones) and to the corresponding scalar one:

$$\gamma_{ii}^{\mathrm{L},p_1p_2p_3} = -\frac{q^2\hbar^2}{800m_0^3c^2\omega_0^2}\left(32\epsilon^2 - 24\epsilon\epsilon_i - 12\epsilon_i^2 + 16\sum_k \epsilon_k^2 + 27\right) \tag{25}$$

$$\gamma_{l=1}^{\mathrm{L},p_1p_2p_3} = -\frac{3q^2\hbar^2}{800m_0^3c^2\omega_0^2}\left(8\epsilon^2 + 4\sum_k \epsilon_k^2 + 9\right) \tag{26}$$

while the 'contact' (Langevin-type) part of the toroid dipole moment induced as a result of interaction in equation (11) is:

$$T_i^{\mathrm{L,ind}} = -\partial\Delta E_p/\partial[(\nabla\times \boldsymbol{H}^{\mathrm{ext}})^{(c)}]_i = \gamma_{ii}^{\mathrm{L},p_1p_2p_3}[(\nabla\times \boldsymbol{H}^{\mathrm{ext}})^{(c)}]_i. \tag{27}$$

For a (spherical symmetric) ground-state equation (26) turns into

$$\gamma_{l=1}^{\mathrm{L},000} = -\frac{9q^2\hbar^2}{80m_0^3c^2\omega_0^2}. \tag{28}$$

The total toroid dipole moment induced in the oscillator will be given by the sum of the moments induced as a result of the perturbations in equations (10) and (11). For a time-independent external perturbation it is characterized by the total toroid dipole polarizability

$$\gamma_{ii}^{\mathrm{T},p_1p_2p_3}(\omega=0) = \gamma_{ii}^{p_1p_2p_3}(\omega=0) + \gamma_{ii}^{\mathrm{L},p_1p_2p_3} = \frac{q^2\hbar^2}{48m_0^3c^2\omega_0^2}\left(4\epsilon\epsilon_i - 2\epsilon_i^2 - 2\sum_k \epsilon_k^2 - 3\right) \tag{29}$$

$$\gamma_{l=1}^{\mathrm{T},p_1p_2p_3}(\omega=0) = \gamma_{l=1}^{p_1p_2p_3}(\omega=0) + \gamma_{l=1}^{\mathrm{L},p_1p_2p_3} = \frac{q^2\hbar^2}{144m_0^3c^2\omega_0^2}\left(4\epsilon^2 - 8\sum_k \epsilon_k^2 - 9\right). \tag{30}$$

For $p_1 = p_2 = p_3 = 0$ (ground-state oscillator)

$$\gamma_{l=1}^{\mathrm{T},000}(\omega=0) = -\frac{q^2\hbar^2}{24m_0^3c^2\omega_0^2}. \tag{31}$$

4. Comparison between induced toroid effects and the usual electric and magnetic ones

In connection with the elastic scattering of low-energy photons on an arbitrary system (Compton scattering), as shown in [6, 8] (see also [7]), unlike the usual (static) electric and magnetic dipole ($l = 1$) polarizabilities $\alpha_{l=1}(\omega = 0)$ and $\beta_{l=1}(\omega = 0)$ which establish the angular structure of the amplitude to the second order in the light frequency ω, the static toroid dipole polarizability $\gamma_{l=1}(\omega = 0)$ enters only at the beginning with the next relevant (fourth) ω order together with the usual (also static) electric and magnetic quadrupole polarizabilities $\alpha_{l=2}(\omega = 0)$, $\beta_{l=2}(\omega = 0)$ and the derivative of the usual (dynamic) electric dipole polarizability $\alpha_{l=1}{}'(\omega = 0) = [\mathrm{d}\alpha_{l=1}(\omega)/\mathrm{d}\omega^2]_{\omega=0}$ and also with the Langevin-type (contact) magnetic and toroid polarizabilities.

For comparison we give the expressions for the electric and magnetic polarizabilities of the same quantum system we used to obtain the toroid polarizability (i.e. the charged tridimensional oscillator) maintaining the same notation we used previously.

The dynamic electric dipole and quadrupole polarizabilities for an arbitrary quantum system whose energy spectrum is that of a tridimensional oscillator are:

$$\alpha_{l=1}^{p_1p_2p_3}(\omega) = \frac{q^2}{m_0(\omega_0^2 - \omega^2)} \tag{32}$$

$$\alpha_{l=2}^{p_1p_2p_3}(\omega) = \frac{2q^2\hbar\epsilon}{m_0^2\omega_0(4\omega_0^2 - \omega^2)}. \tag{33}$$

The low-frequency limit gives the static electric dipole and quadrupole polarizabilities and the derivative of the dynamic electric dipole polarizability:

$$\alpha_{l=1}^{p_1p_2p_3}(\omega) = \alpha_{l=1}^{p_1p_2p_3}(\omega = 0) + \frac{\omega^2}{c^2}\alpha_{l=1}^{p_1p_2p_3\prime}(\omega = 0) + \cdots \tag{34}$$

$$\alpha_{l=1}^{p_1p_2p_3}(\omega = 0) = \frac{q^2}{m_0\omega_0^2} \tag{35}$$

$$\alpha_{l=1}^{p_1p_2p_3\prime}(\omega = 0) = c^2[\mathrm{d}\alpha_{l=1}^{p_1p_2p_3}(\omega)/\mathrm{d}\omega^2]_{\omega=0} = \frac{q^2c^2}{m_0\omega^4} \tag{36}$$

$$\alpha_{l=2}^{p_1p_2p_3}(\omega = 0) = \frac{q^2\hbar\epsilon}{2m_0^2\omega_0^3}. \tag{37}$$

The dynamic magnetic dipole polarizability for the isotropic oscillator $\beta_{l=1}^{p_1p_2p_3}(\omega)$ vanishes, but there is a Langevin-type one (responsible for the Langevin diamagnetism) which has a non-vanishing form:

$$\beta_{l=1}^{\mathrm{L},p_1p_2p_3} = -\frac{q^2\hbar\epsilon}{6m_0^2\omega_0c^2}. \tag{38}$$

The dynamic magnetic quadrupole polarizability and the corresponding total static one (including the Langevin-type piece) are:

$$\beta_{l=2}^{p_1p_2p_3}(\omega) = \frac{q^2\hbar^2}{6m_0^3c^2}\frac{2\epsilon^2 - 2\sum_k \epsilon_k^2 - 3}{\omega_0^2 - \omega^2} \tag{39}$$

$$\beta_{l=2}^{\mathrm{T},p_1p_2p_3}(\omega = 0) = \frac{q^2\hbar^2}{20m_0^3c^2\omega_0^2}\left(4\epsilon^2 - 8\sum_k \epsilon_k^2 - 13\right). \tag{40}$$

By comparison with $\alpha_{l=1}^{p_1p_2p_3}(\omega)$, $\alpha_{l=2}^{p_1p_2p_3}(\omega)$ and $\beta_{l=2}^{p_1p_2p_3}(\omega)$ in equations (32), (33) and (39) it can be seen that $\gamma_{l=1}^{p_1p_2p_3}(\omega)$ in equation (17) has an additional resonance frequency $\omega_{\mathrm{res}} = 3\omega_0$ (ω_0 being the frequency of the oscillator). This resonance frequency can only appear at the beginning with the octupole polarizability in the electric case ($\omega_{\mathrm{res}} = 3\omega_0$ is a resonance frequency for $\alpha_{l=2n+1}$ with $n \geqslant 1$), but this polarizability does not have the same dimension and importance as $\alpha_{l=2}^{p_1p_2p_3}(\omega)$, $\beta_{l=2}^{p_1p_2p_3}(\omega)$ and $\gamma_{l=1}^{p_1p_2p_3}(\omega)$ (it contains in addition the square of the system's dimension).

So, the dynamic toroid dipole polarizability is the only one of the dipole and quadrupole classes of polarizabilities which has a resonance frequency $\omega_{\mathrm{res}} = 3\omega_0$. Consequently, for an external field interaction whose frequency is close to $3\omega_0$ the induced toroid dipole moments and structure (characterized by the ability of the constituent charges of the system to move on eight-like orbits) become predominant over the other induced electric and magnetic moments. This could lead to new effects.

In the static case a comparison between $\gamma_{l=1}^{p_1p_2p_3}(\omega = 0)$, $\alpha_{l=2}^{p_1p_2p_3}(\omega = 0)$, $\alpha_{l=1}^{p_1p_2p_3\prime}(\omega = 0)$ and $\beta_{l=2}^{T,p_1p_2p_3}(\omega = 0)$ can be made. Using equations (30), (36), (37) and (40) the following exact formulae are obtained:

$$\frac{\gamma_{l=1}^{\mathrm{T},p_1p_2p_3}(\omega=0)}{\alpha_{l=1}^{p_1p_2p_3}{}'(\omega=0)} = \frac{1}{144}\left(\frac{\hbar\omega_0}{m_0c^2}\right)^2\left(4\epsilon^2 - 8\sum_k \epsilon_k^2 - 9\right) \tag{41}$$

$$\frac{\gamma_{l=1}^{\mathrm{T},p_1p_2p_3}(\omega=0)}{\alpha_{l=2}^{p_1p_2p_3}(\omega=0)} = \frac{1}{72}\left(\frac{\hbar\omega_0}{m_0c^2}\right)\frac{(4\epsilon^2 - 8\sum_k \epsilon_k^2 - 9)}{\epsilon} \tag{42}$$

$$\frac{\gamma_{l=1}^{T,p_1p_2p_3}(\omega=0)}{\beta_{l=2}^{\mathrm{T},p_1p_2p_3}(\omega=0)} = \frac{5}{36}\frac{4\epsilon^2 - 8\sum_k \epsilon_k^2 - 9}{4\epsilon^2 - 8\sum_k \epsilon_k^2 - 13}. \tag{43}$$

For $\hbar\omega_0/m_0c^2 \ll 1$ one can see that the effects due to the induced toroid moments appear to be very small compared with those of the corresponding usual electric ones (although they increase with the oscillator's energy) and of the same order of magnitude as those related to the induced quadrupole magnetic moments. The same conclusion arises when considering a (non-relativistic, spinless, ground-state) hydrogen-like atom [7]. This could be one of the reasons why induced toroid moments have so far been disregarded in atomic physics.

In the field of elementary particles physics a similar comparative analysis can be made considering a typical example of hadron, the (charged) pion [6, 11]. (The example of the charged pion is not only of pure academic interest. The Compton effect on the charged pion has already been experimentally observed and studied and the electric dipole polarizability of the charged pion has been experimentally extracted in two different experiments [12].)

In [6] an order of magnitude estimate of the static toroid dipole polarizability of $\pi^\pm$ has been obtained (by evaluating the A_1(1270 MeV) meson resonance contribution to $\gamma_{l=1}^{\pi^\pm}(\omega = 0)$ in terms of the experimentally known radiative width $\Gamma(A_1 \rightarrow \pi\gamma) \simeq 0.6$ MeV) with the result (see also [7]) $\gamma_{l=1}^{\pi^\pm}(\omega = 0) \simeq 1.2 \times 10^{-5}$ fm^5. Under the same aproximations it has been found [6] (see also [7]) that $\alpha_{l=1}^{\pi^\pm}{}'(\omega = 0) \simeq 0.8 \times 10^{-5}$ fm^5. The static electric and magnetic quadrupole polarizability of $\pi^\pm$ are expected to be of the order [11] $\alpha_{l=2}^{\pi^\pm}(\omega = 0) \sim \beta_{l=2}^{\pi^\pm}(\omega = 0) \sim 10^{-5}$ fm^5. It has been found that

$$\frac{\gamma_{l=1}^{\pi^\pm}(\omega=0)}{\alpha_{l=1}^{\pi^\pm}{}'(\omega=0)} \sim \frac{\gamma_{l=1}^{\pi^\pm}(\omega=0)}{\alpha_{l=2}^{\pi^\pm}(\omega=0)} \sim \frac{\gamma_{l=1}^{\pi^\pm}(\omega=0)}{\beta_{l=2}^{\pi^\pm}(\omega=0)} \sim 1. \tag{44}$$

The result is in sharp contrast to the corresponding one from the atomic physics. For hadrons, the toroid polarizabilities (and the related effects) are expected to be of the same order of magnitude as the usual electric and magnetic ones (of one order of multipolarity higher).

Equations (41)–(43) can provide a theoretical explanation for the conclusion [6, 7] that the more 'elementary' the object is (or the higher are the characteristic excitation energies of the system) the better it responds to an external (conduction and/or displacement) current $\nabla \times \boldsymbol{H}^{\mathrm{ext}}$ rather than to the electric $\boldsymbol{E}^{\mathrm{ext}}$ and magnetic $\boldsymbol{H}^{\mathrm{ext}}$ fields directly, because the importance of the induced toroid effects increases (in the oscillator model) with E/m_0c^2.

Toward smaller distances (at the subhadronic level) the induced toroid effects might increase further and become predominant over the usual (induced) electric ones. One possible explanation may be the fact that for a classical toroidal current (of large and small radii of the toroidal solenoid $R_\mathrm{T}, r_\mathrm{T}$) with N turns of winding and a current intensity I, the classical toroid dipole moment is [2, 6]

$$\boldsymbol{T}_{\mathrm{Torus}} = \boldsymbol{n}\frac{NIV_\mathrm{T}}{4\pi c} \tag{45}$$

where V_T is the volume of the torus and n is the unit vector along the axis of the toroid. The induced toroid moment increases on account of the large number of turns in the winding, so the systems which have filiform structure (strings) could have a large toroid polarizability [6, 7].

Because the single possible intrinsic electromagnetic structure of the Majorana fermions is required to consist only of toroid moments and distribution [5], these particles would probably have large toroid polarizabilities just as macroscopic substances composed of polar molecules (i.e. molecules with an intrinsic electric dipole moment) would generally have a large electric polarizability [7]. For the same reason, atoms, molecules and nuclei having intrinsic toroid moments (as those in [3, 4]) could possess important toroid polarizabilities.

5. Temperature's influence on the induced toroid effects

If the oscillator is in thermodynamic equilibrium, the mean toroid dipole polarizabilities can be calculated with the Gibbs distribution. The result for the dynamic polarizability, from equation (17), has the following dependence on the system's temperature (T):

$$\overline{\gamma_{l=1}(\omega)} = \frac{q^2\hbar^2}{160m_0^3c^2}\left[\frac{3(3\coth^2\hbar\omega_0/2kT + 1)}{9\omega_0^2 - \omega^2} + \frac{17\coth^2\hbar\omega_0/2kT - 7}{\omega_0^2 - \omega^2}\right] \tag{46}$$

(k denotes Boltzmann's constant). For $\omega = 0$ the static form of the mean toroid dipole polarizability is obtained:

$$\overline{\gamma_{l=1}(\omega = 0)} = \frac{q^2\hbar^2}{240m_0^3c^2\omega_0^2}\left(27\coth^2\frac{\hbar\omega_0}{2kT} - 10\right). \tag{47}$$

Adding the mean Langevin-type contribution obtained from equation (26),

$$\overline{\gamma_{l=1}^{\mathrm{L}}} = -\frac{9q^2\hbar^2}{80m_0^3c^2\omega_0^2}\coth^2\frac{\hbar\omega_0}{2kT} \tag{48}$$

at $\overline{\gamma_{l=1}(\omega = 0)}$ from equation (47) one gets the mean toroid dipole polarizability:

$$\overline{\gamma_{l=1}^{\mathrm{T}}(\omega = 0)} = -\frac{q^2\hbar^2}{24m_0^3c^2\omega_0^2}. \tag{49}$$

It can be seen that $\overline{\gamma_{l=1}^{\mathrm{T}}(\omega = 0)}$ has no dependence on temperature and the mean total toroid dipole moment induced in the system by an external field configuration which is constant in time has no classical analogue.

For a time-dependent external interaction ($\omega \neq 0$) and for a temperature $T \gg \hbar\omega_0/k$ equation (46) may be written in a form that cancels the $\hbar^2$ factor and permits a classical interpretation:

$$\overline{\gamma_{l=1}(\omega)} \simeq \frac{q^2k^2T^2}{40m_0^3c^2\omega_0^2}\left(\frac{9}{9\omega_0^2 - \omega^2} + \frac{17}{\omega_0^2 - \omega^2}\right). \tag{50}$$

A comparison with the other dynamic electric and magnetic polarizabilities can also be made. Since $\alpha_{l=1}(\omega)$ has no dependence on the quantum numbers p_1, p_2, p_3 it has no dependence on temperature either. The (dynamic) electric and magnetic quadrupole polarizabilities have the following mean values:

$$\overline{\alpha_{l=2}(\omega)} = \frac{3q^2\hbar}{m_0^2\omega_0(4\omega_0^2 - \omega^2)}\coth\frac{\hbar\omega_0}{2kT} \tag{51}$$

$$\overline{\beta_{l=2}(\omega)} = \frac{q^2\hbar^2}{2m_0^3c^2(\omega_0^2 - \omega^2)}\left(\coth^2\frac{\hbar\omega_0}{2kT} - 1\right). \tag{52}$$

For $T \gg \hbar\omega_0/k$ equations (51) and (52) turn into

$$\overline{\alpha_{l=2}(\omega)} \simeq \frac{6q^2kT}{m_0^2\omega_0^2(4\omega_0^2-\omega^2)} \tag{53}$$

$$\overline{\beta_{l=2}(\omega)} \simeq \frac{2q^2k^2T^2}{m_0^3c^2\omega_0^2(\omega_0^2-\omega^2)}. \tag{54}$$

So, from equations (50) and (53), even if the (time-dependent) induced toroid dipole moments are smaller than the quadrupole electric ones, their relative importance, however, increases with T. Comparing the dynamic toroid effects with the quadrupole magnetic ones in equation (54) one can see that they have the same temperature dependence.

Analysing equations (50), (53) and (54) one can see that the mean polarizabilities maintain the same resonance frequencies as those which characterize an arbitrary state system, in equations (17), (33) and (39).

For a static external field, a comparison between induced toroid, electric and magnetic effects leads to

$$\overline{\gamma_{l=1}^{\mathrm{T}}(\omega=0)}/\overline{\alpha_{l=1'}(\omega=0)} = -\frac{1}{24}\left(\frac{\hbar\omega_0}{m_0c^2}\right)^2 \tag{55}$$

$$\overline{\gamma_{l=1}^{\mathrm{T}}(\omega=0)}/\overline{\alpha_{l=2}(\omega=0)} = -\frac{1}{18}\frac{\hbar\omega_0}{m_0c^2\coth\hbar\omega_0/2kT} \simeq -\left(\frac{\hbar^2\omega_0^2}{36m_0c^2kT}\right)_{(T\gg\hbar\omega_0/k)} \tag{56}$$

$$\overline{\gamma_{l=1}^{\mathrm{T}}(\omega=0)}/\overline{\beta_{l=2}^{\mathrm{T}}(\omega=0)} = -\tfrac{1}{12}. \tag{57}$$

6. Conclusions

Although the induced *static* toroid effects can hardly be highlighted in atomic and solid state physics (this could be one of the reasons for neglecting the toroid moments in these domains), the *dynamic* ones can be observed on account of their different dependence on the external field frequency (in the charged oscillator model an additional specific resonance frequency appears).

While for atoms (length scale 10^{-8} cm) the static toroid polarizabilities are generally negligible quantities, for hadrons (length scale 10^{-13} cm) they can no longer be neglected as one expects them to be of the same order of magnitude as the corresponding usual electric and magnetic ones. For more 'elementary' objects the importance of the induced moments is expected to increase further.

Acknowledgments

The author is indebted to Professor E E Radescu for an introduction to the field of toroid moments and for help, advice and patience.

References

[1] Zel'dovich Ya B 1957 *Zh. Eksp. Teor. Fiz.* **33** 1531 (Engl. Transl. 1958 *Sov. Phys.–JETP* **6** 1184)
[2] Dubovik V M and Tcheshkov A A 1974 *Fiz. Elem. Chastits At. Yadra* 5 791 (Engl. Transl. *Sov. J. Part. Nucl.* 5 318)
Dubovik V M and Tosunian L A 1983 *Fiz. Elem. Chastits At. Yadra* **14** 1193 (Engl. Transl. 1983 *Sov. Phys.–JETP* **14** 504)
Dubovik V M and Tugushev V V 1990 *Phys. Rep.* **187** 145

[3] Khriplovich I B 1991 *Parity Nonconservation in Atomic Phenomena* (New York: Gordon and Breach) ch 8
Apenko S M and Lozovik Y E 1982 *J. Phys. B: At. Mol. Phys.* **15** L57
Khriplovich I B and Pospelov M E 1990 *Z. Phys.* D **17** 81
Boston E R and Sandars P G H 1990 *J. Phys. B: At. Mol. Opt. Phys.* **23** 2663
Lewis R R 1993 *Phys. Rev.* A **48** 4107; 1994 *Phys. Rev.* A **49** 3376

[4] Flambaum V V and Khriplovich I B 1980 *Zh. Eksp. Teor. Fiz.* **79** 1656 (Engl. Transl. *Sov. Phys.–JETP* **52** 835)
Sushkov O P, Flambaum V V and Khriplovich I B 1984 *Zh. Eksp. Teor. Fiz.* **87** 1521 (Engl. Transl. *Sov. Phys.–JETP* **60** 873)
Flambaum V V, Khriplovich I B and Sushkov O P 1984 *Phys. Lett.* **146B** 367; 1986 *Phys. Lett.* **162B** 213; 1986 *Nucl. Phys.* A **449** 750
Haxton W C, Henley E M and Musolf M J 1989 *Phys. Rev. Lett.* **63** 949

[5] Radescu E E 1985 *Phys. Rev.* D **32** 1266

[6] Radescu E E 1985 *Joint Institute for Nuclear Research, Dubna Communication* E4-85-165 unpublished; 1985 *Joint Institute for Nuclear Research, Dubna Report* P4-85-154, P4-85-155, P4-85-156 (1986 *Rev. Roum. Phys.* **31** 139, 143, 145)

[7] Costescu A and Radescu E E 1987 *Phys. Rev.* D **35** 3496; 1991 *Ann. Phys., NY* **209** 13

[8] Radescu E E *Preprint* ICTP, Trieste IC/90/315, IC/90/319 unpublished

[9] Radescu E E unpublished

[10] Vlad D H unpublished

[11] Guiasu I and Radescu E E 1979 *Ann. Phys., NY* **120** 145; 1979 *Ann. Phys., NY* **122** 436; 1982 *Phys. Rev.* D **25** 1455
Guiasu I, Pomponiu C and Radescu E E 1978 *Ann. Phys., NY* **114** 296
Radescu E E 1978 *Phys. Rev.* D **18** 2329
Filkov L V, Guiasu I and Radescu E E 1982 *Phys. Rev.* D **26** 3146

[12] Aibergenov T A *et al* 1982 *Kratk. Soobshch. Fiz.* **5** 33; 1984 *Kratk. Soobshch. Fiz.* **6** 31; 1986 *Czech. J. Phys.* B **36** 948
Antipov Yu M *et al* 1983 *Phys. Lett.* **121B** 445 (1983); 1985 *Z. Phys.* C **26** 495
Kowalevski R V *et al* 1984 *Phys. Rev.* D **29** 1000